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ON THE THEORY OF CONTACT PROBLEMS TAKING ACCOUNT OF FRICTION ON THE CONTACT SURFACE*

A. S. KRAVCHUK

Problems of contact of elastic bodies, taking account of friction on the contact surface, are investigated. A new formulation is constructed, which is based on reducing the problem to some nonclassical problem of the calculus of variations. Algorithms are proposed for the construction of approximate solutions and their convergence is given a foundation. Two friction laws, namely those of Coulomb and Prandtl—Il'iushin, are examined.

1. Differential formulation of the quasistatic problem. The problem of the contact between a linearly elastic body and an absolutely rigid stamp will be examined in detail; possible generalizations are obtained on the basis of results in the paper /1/ (nonlinear elasticity, several bodies in contact).

We give the equation of the stamp surface in the form

$$\Psi(x) = 0$$

(see /2/ for hypotheses relative to the function $\Psi(x)$.

The complete system of equations and the boundary conditions describing the process of inserting a stamp with friction into an elastic solid is (this formulation is possible when the stamp displacement is given; the problem is written in a coordinate system coupled rigidly to the stamp; the case of assigning the forces is given in /1/):

$$-\nabla\cdot({}^4a\cdot\cdot\,\varepsilon(u)) = \rho F \tag{1.1}$$

$$\sigma\cdot\nu\,|_{S_\sigma} = P \tag{1.2}$$

$$u\,|_{S_u} = g \tag{1.3}$$

$$\gamma(x) \equiv \Psi(x) + u(x)\cdot\nabla\Psi(x) \geqslant 0, \quad \forall x \in S_c \tag{1.4}$$

$$\gamma(x) > 0 \Rightarrow \sigma\cdot\nu = 0 \tag{1.5}$$

$$\gamma(x) = 0 \Rightarrow \sigma_N \equiv (\sigma\cdot\nu)\cdot\nu \leqslant 0 \tag{1.6}$$

$$|\sigma_T| \equiv |\sigma\cdot\nu - \sigma_N\nu| < f\,|\sigma_N| \Rightarrow |u_T^{\cdot}| = 0$$

$$|\sigma_T| = f\,|\sigma_N| \Rightarrow \frac{u_T^{\cdot}}{|u_T^{\cdot}|} = -\frac{\sigma_T}{|\sigma_T|}, \quad x \in S_c, \quad S_\sigma \cup S_u \cup S_c = S$$

Here (1.1) is the equilibrium equation, ∇ is the Hamilton operator, ${}^4a$ is the elastic modulus tensor, $\varepsilon(u)$ is the strain tensor, u is the displacement vector, ρF is the volume force vector, σ is the stress tensor, ν is the normal to S the boundary of the deformable body occupying a domain Ω, P are surface loads, and g are given displacements.

The last group of relationships in (1.6) express the Coulomb friction law: f is the friction coefficient, $u_T^{\cdot}$ is the velocity of body particle motion over the stamp in a projection on the tangent plane, defined as the derivative with respect to the parameter t giving the process of the change in external actions, $t \in [0, T]$.

Therefore, the solution $u = u(x, t)$ is a function of the spatial coordinates and the parameter t; the problem is to determine $u(x, t)$ which will satisfy the system of equations and the conditions (1.1)—(1.6), where the set of points $x \in S_c$, for which $\gamma(x) = 0$ is the contact zone the set of points $S_{c^*} \subset S_c$ for which $|\sigma_T| < f|\sigma_N|$ is the adhesion zone, and the vector $\sigma\cdot\nu$ on S_c is the contact interaction force, should be determined in the process of the solution. The state of the body at $t = 0$ is considered unstressed and unstrained: $u(x, 0) = u^0(x) = 0$.

2. Derivation of the quasivariational inequality. Interpretation. Let us first note that the relations (1.1)—(1.6) permit the determination of just the velocity $u^{\cdot}$ (or, equivalently, the first differential of the solution $du^t(x, t)$ at the time t). Let u^t denote the solution at the time t and let us construct an inequality relating u^t, $u^{t+dt} \equiv u^t + du^t$.

Let us form the invariant

$$E(\sigma^t, v) \equiv E^t(v) = \int_\Omega \sigma^t\cdot\cdot\,\varepsilon(v)\,d\Omega - L_*^{\,t}(v) - \tag{2.1}$$

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$$\int_{S_c} (\sigma^t\cdot\nu)\cdot v\,dS,\quad L_*{}^t(v)=\int_\Omega \rho F^t\cdot v\,d\Omega+\int_{S_\sigma} P^t\cdot v\,dS$$

where σ^t is the true stress field at the time t; evidently $E^t(v)=0,\ \forall v$ that follows from the equilibrium condition of the body Ω at the time t.

Let u^{t+dt} be the solution for the time $t+dt$, w the kinematically possible state satisfying the support condition on S_v, the nonpenetration condition on S_c, but not necessarily the last group of relationships in (1.6) (the Coulomb law).

Let us form the difference

$$\Delta E=E^{t+dt}(w)-E^t(u^t)-[E^{t+dt}(u^{t+dt})-E^t(u^t)] \tag{2.2}$$

It can be established that

$$\Delta E=a(u^{t+dt},w-u^{t+dt})-L_*^{t+dt}(w-u^{t+dt})-\int_{S_c}(\sigma^{t+dt}\cdot\nu)\cdot(dw-du^t)\,dS \tag{2.3}$$

$$\left(a(u,v)=\int_\Omega \sigma(u)\cdot\cdot\,\varepsilon(v)\,d\Omega,\quad dw=w-u^t\right)$$

The following estimate holds

$$[\sigma(u)\cdot\nu]\cdot(dw-du^t)\geqslant f\,|\,\sigma_N(u)\,|\,(|\,dw_T\,|-|\,du_T\,|) \tag{2.4}$$

Therefore, the solution of the problem (1.1)—(1.6) satisfies the inequality

$$a(u^{t+dt},w-u^{t+dt})-L_*^{t+dt}(w-u^{t+dt})+\int_{S_c} f\,|\sigma_N^{t+dt}|\,(|\,dw_T\,|-|\,du_T\,|)\,dS\geqslant 0,\quad \forall w \tag{2.5}$$

Inequalities of the type (2.5) in /3/ are called quasivariational.

The following theorem holds: the solution of the inequality (2.5) that possesses generalized second derivatives will satisfy all the inequalities and conditions (1.1)—(1.6). The proofs of the estimate (2.4) and the mentioned theorem are cumbersome and will be published separately.

3. Relation to the method of local potential.

Let us form the functional

$$J(u^{t+dt},v)=0.5a(v,v)-L_*^{t+dt}(v)+\int_{S_c} f\,|\sigma_N(u^{t+dt})|\,|\,v_T-u_T{}^t\,|\,dS \tag{3.1}$$

and let us examine the following conditional extremum problem

$$J(u^{t+dt},u^{t+dt})\leqslant J(u^{t+dt},v),\quad \forall v\in K \tag{3.2}$$

$$u^{t+dt}\in K,\quad u^{t+dt}=v$$

$$K=\{v\mid \Psi(x)+v(x)\cdot\nabla\Psi(x)\geqslant 0,\ x\in S_c\}$$

The problem (3.2) is understood as follows: An element u^{t+dt} is fixed and the following absolute minimization problem is solved

$$J(u^{t+dt},v)\to\min\quad \text{on } K$$

whereupon the element u^* is determined; it is necessary to select a u^{t+dt} such that the equality $u^{t+dt}=u^*$ would hold. This is a problem of the local potential method of I. Prigogin /4/ (see /5/ also).

By compiling the necessary conditions for the minimum of the functional (3.1) (taking its nondifferentiability into account), we arrive at the inequality (2.5), therefore, the local potential technique /4,5/ can be applied to solve the problem under consideration in this paper. Unfortunately, the state of the art in this method does not permit the effective construction of a solution and obtaining a foundation to prove existence and uniqueness theorems, hence, another method will be used below which is borrowed from the theory of quasivariational inequalities.

4. Method of successive approximations and existence theorems.

To solve the inequality (2.5) we use the following iteration process (we omit the superscript $t+dt$):

$$a(u^{r+1},v-u^{r+1})-L_*(v-u^{r+1})+\int_{S_c} f\,|\,\sigma_N(u^r)\,|\,(|\,v_T-u_T{}^t\,|-|\,u_T^{r+1}-u_T{}^t\,|)\,dS\geqslant 0 \tag{4.1}$$

$$\forall v\in K,\quad u^{r+1}\in K$$

The quantity u° is given, and r is the number of the iteration. For each value of the superscript r the inequality (4.1) reduces to an ordinary problem of minimizing the functional

$$J(v) = 0{,}5a(v, v) - L_*(v) + \int_{S_c} f\,|\sigma_N(u^r)|\,|v_T - u_T{}^t|\,dS \tag{4.2}$$

in the set K; the assertion about the existence and uniqueness of the solution u^{r+1} follows rapidly from classical theorems on the existence and uniqueness of the minimum of a strictly convex functional in a closed convex set. The following (restricted) result can be established relative to the existence of a solution: if the friction coefficient f is sufficiently small (see below), then the sequence defined by the process (4.1) will reduce to the solution of the problem (2.5).

Proof. Let us write the inequality (4.1) for the preceding step of the iteration process, we put $v = u^{r+1}$ in the inequality obtained, we put $v = u^r$ in the inequality (4.1), and we add the results. We find

$$\begin{gathered} -a(u^{r+1} - u^r, u^{r+1} - u^r) + \int_{S_c} [f\,|\sigma_N{}^r| - f\,|\sigma_N^{r-1}|] \times \\ [|u_T{}^r - u_T{}^t| - |u_T^{r+1} - u_T{}^t|]\,dS \geqslant 0 \end{gathered} \tag{4.3}$$

An inequality of positive definiteness holds

$$a(v, v) \geqslant \alpha\,\|v\|^2_{H^1\Omega},\ \alpha = \text{const} > 0 \tag{4.4}$$

By using a theorem on traces /6/, we establish that

$$\int_{S_c} |\sigma_N(u)|\,|v_T|\,dS \leqslant C\,\|u\|_{H^1(\Omega)}\,\|v\|_{H^1(\Omega)} \tag{4.5}$$

where C is a constant defined by the elastic moduli, the body shape, and the fixing conditions.

Introducing the notation $\delta u^r = u^r - u^t$ and using the inequalities (4.4) and (4.5), we establish that

$$\|\delta u^{r+1} - \delta u^r\|_{H^1(\Omega)} \leqslant \frac{Cf}{\alpha}\,\|\delta u^r - \delta u^{r-1}\|_{H^1(\Omega)}$$

Now, let us require that the friction coefficient f satisfy the inequality $f < \alpha / C$. Then the sequence $\{u^r\}$ defined by the process (4.1) will converge; it is established in the usual manner that its limit will satisfy the inequality (2.5).

Let us note that the theoretical calculation of the upper bound α / C for f is a difficult problem.

5. Algorithm for the practical solution of the problem based on the idea of duality. By using the results of /7/, we reduce the problem (4.2) first to the form

$$\begin{gathered} \min_{v \in V} \max_{\sigma_N \leqslant 0} \Big[J(v) + \int_{S_c} \sigma_N(\delta_N - v_N)\,dS\Big] \\ \delta_N = \Psi(x) \,/\, |\nabla\Psi(x)| \end{gathered} \tag{5.1}$$

Let us note that

$$f\,|\sigma_N{}^r|\,|v_T - u_T{}^t| = \max_{\mu_T,\ |\mu_T| \leqslant f|\sigma_N{}^r|} \mu_T \cdot (v_T - u_T{}^r) \tag{5.2}$$

Therefore, the problem (5.1) reduces to the following problem of seeking a saddle point

$$\begin{gathered} \min_{v \in V} \max_{\sigma_N \leqslant 0} \max_{|\mu_T| \leqslant f|\sigma_N{}^r|} \times \\ \Big\{0.5a(v, v) - L_*(v) + \int_{S_c} [\mu_T \cdot (v_T - u_T{}^t) + \sigma_N(\delta_N - v_N)]\,dS\Big\} \end{gathered} \tag{5.3}$$

Using the condition of stationarity of the functional, it can be shown that $\mu_T = -\sigma_T$. The solution of the problem (5.3) was carried out by using an algorithm of Udzawa type (Arrow—Hurwitz), which here takes the form:

1) Distributions of $\sigma_N^{r+1,0}$, $\sigma_T^{r+1,0}$ on S_c are given;

2) The problem (5.3), which is equivalent to the usual elasticity theory problem with a boundary condition on S_C of the form $\sigma_{ij}\nu_j = \sigma_N^{r+1,0}\nu_i + (\sigma_T^{r+1,0})_i$, is solved;

3) New distributions of the contact interaction forces are constructed by means of the formulas

$$\begin{gathered} \sigma_N^{r+1,1} = P_N\,[\sigma_N^{r+1,0} + \rho_{0N}(\delta_N - u_N^{r+1,0})] \\ \sigma_T^{r+1,1} = P_T\,[\sigma_T^{r+1,0} + \rho_{0T}(u_T^{r+1,0} - u_T{}^t)] \end{gathered}$$

where

$$P_N(h) = \begin{cases} 0, & h > 0 \\ h, & h \leqslant 0 \end{cases}, \quad P_T(l) = \begin{cases} l, & |l| \leqslant f|\sigma_N^0| \\ \frac{1}{|l|} lf|\sigma_N^0|, & |l| > f|\sigma_N^0| \end{cases}$$

are orthogonal projection operators on the set $\sigma_N \leqslant 0$, $|\sigma_T| \leqslant f|\sigma_N^r|$, ρ_{0N}, ρ_{0T} are parameters controlling the convergence of the method.

The classical algorithm of the method of quasivariational inequalities is that for a fixed σ_T the iterations on σ_N are performed until convergence is achieved.

The convergence of the algorithm 1) — 3) is assured by the concave-convex structure of the problem (strict convexity in v and concavity in σ_N, σ_T).

6. Dynamic problem. Let us consider this problem on the basis of the principle of admissible velocities; in conformity with this principle, we have

$$\langle \rho u^{\cdot\cdot}, \delta u^{\cdot} \rangle + a(u, \delta u^{\cdot}) = \int_{S_c} \sigma(u) \cdot \mathbf{v} \cdot \delta u^{\cdot} dS + L_*(\delta u^{\cdot}) \tag{6.1}$$

where

$$\langle \rho u^{\cdot\cdot}, \delta u^{\cdot} \rangle = \int_{\Omega} \rho \frac{\partial^2 u}{\partial t^2} \cdot \frac{\partial \delta u}{\partial t} d\Omega, \quad \delta u^{\cdot} = v^{\cdot} - u^{\cdot}$$

($u^{\cdot}$ is the true velocity field, $v^{\cdot}$ is the kinematically admissible velocity field; the functionals a and L_* are defined in Sect.2).

Generalizing the Ostrogradskii construction /8/ (with the Mayer and Zermelo refinements) to the case of the continual one-sided relation available here, we introduce the set of points

$$S_c^t = \{x \mid x \in S_c;\ \Psi(x) + u(x, t) \cdot \nabla\Psi(x) = 0$$
$$u^{\cdot}(x, t) \cdot \nabla\Psi(x) = 0,\ u^{\cdot}(x, t) \cdot \nabla\Psi(x) = 0\}$$

where $u(x, 0)$ and $u^{\cdot}(x, 0)$ are determined by the initial conditions, $u^{\cdot\cdot}(x, 0)$ is determined by using the Gauss principle of least constraint, and we subject the set $\delta u^{\cdot}$ to the constraint

$$\delta u^{\cdot}(x, t + dt) \cdot \nabla\Psi(x) \geqslant 0,\ \forall x \in S_c^t$$

Then we have the following estimate for the velocity of work of the normal pressure

$$\int_{S_c} \sigma_N \delta u_N^{\cdot} dS \geqslant 0 \tag{6.2}$$

The inequality

$$\sigma_T(u) \cdot \delta u_T^{\cdot} \geqslant f \mid \sigma_N(u) \mid (\mid v_T^{\cdot} \mid - \mid u_T^{\cdot} \mid) \tag{6.3}$$

holds, which is established exactly as is the inequality (2.4).

Taking account of (6.2) and (6.3), we conclude that the solution of the dynamic problem satisfies the inequality

$$\langle \rho u^{\cdot\cdot}, \delta u^{\cdot} \rangle + a(u, \delta u^{\cdot}) + \int_{S_c} f|\sigma_N(u)|(|v_T^{\cdot}| - |u_T^{\cdot}|) dS \geqslant \tag{6.4}$$
$$L_*(\delta u^{\cdot}), \quad \delta u^{\cdot} \cdot \nabla\Psi(x) \geqslant 0, \quad \forall x \in S_c^t$$

Conversely, any solution of inequality (6.4) possessing second derivatives will satisfy equations (1.1), taking account of the inertia terms, and the conditions (1.2) — (1.6).

7. A priori estimate of the solution. Let us note that $\sigma_N u_N^{\cdot} = 0$ on S_c, therefore, we can find from (6.1) ($\rho = \text{const}$)

$$\frac{\partial}{\partial t}\left[\frac{\rho}{2}\langle u^{\cdot}, u^{\cdot} \rangle + \frac{1}{2} a(u, u)\right] + \int_{S_c} |\sigma_T||u_T^{\cdot}| dS = L_*^t(u^{\cdot}) \tag{7.1}$$

(the last of the relationships (1.6) was used in constructing (7.1)).

Let us integrate (7.1) between 0 and t, we obtain

$$\frac{\rho}{2}\langle u^{\cdot} \rangle^2 + \frac{1}{2} a(u, u) + \int_0^t \int_{S_c} |\sigma_T(u)||u_T^{\cdot}| dS d\tau = \tag{7.2}$$
$$\frac{\rho}{2}\langle u^{\cdot}(0) \rangle^2 + \frac{1}{2} a(u(0), u(0)) + \int_0^t L_*^{\tau}(u^{\cdot}(\tau)) d\tau, \ \langle u^{\cdot} \rangle^2 = \langle u^{\cdot}, u^{\cdot} \rangle$$

The following estimate is valid

$$\int_0^t L_*^\tau(u^\cdot)\,d\tau \leqslant \frac{1}{2\varepsilon}\int_0^t \|L_*^\tau\|^2 d\tau + \frac{\varepsilon}{2}\int_0^t \langle u^\cdot\rangle^2 d\tau \tag{7.3}$$

We use the inequality (4.4) of the coercivity, we select the $\min\{\rho/2, \alpha/2\} \equiv c_0$, $\varepsilon/2 = c_0$, we discard the last term on the left in (7.2), and we add the following integral to (7.2) on the right

$$\frac{1}{2}c_0\int_0^t \|u(\tau)\|^2 d\tau$$

and finally obtain the inequality

$$c_0\varphi(t) \leqslant c_1c_0 + \int_0^t \frac{1}{2\varepsilon}\|L_*^\tau\|^2 d\tau + c_0\int_0^t \varphi(\tau)\,d\tau \tag{7.4}$$

$$(\varphi(t) \equiv \langle u^\cdot(t)\rangle^2 + \|u(t)\|^2, \quad c_1c_0 = \frac{\rho}{2}\langle u^\cdot(0)\rangle^2 + \frac{1}{2}a(u(0), u(0)))$$

By using the Grunwall inequality we obtain from (7.4)

$$\varphi(t) \leqslant \left(\frac{1}{4c_0^2}\int_0^t \|L_*^\tau\|^2 d\tau + c_1\right)e^t \tag{7.5}$$

from which it follows that

$$u(t) \in L^2(0, T; V), \ u^\cdot(t) \in L^2(0, T; L^2(\Omega))$$

(the definition of the space $L^2(0, T; X)$ is found in the book /9/, say, while V is defined exactly as in /2/).

8. Difference schemes for the solution of the dynamic problem. Problem of convergence. Let us present one of the difference schemes (in t) for the solution of the inequality (6.4) (τ is the partition spacing)

$$a(u^{t+\tau}, v - u^{t+\tau}) + L_*^{t+\tau}(v - u^{t+\tau}) + \int_{S_c} f|\sigma_N^{t+\tau}|(|v_T - u_T^t| - |u_T^{t+\tau} - u_T^t|)\,dS + \frac{\rho}{\tau^2}\int_\Omega (u^{t+\tau} - 2u^t + u^{t-\tau})\cdot(v - u^{t+\tau})\,d\Omega \geqslant 0 \tag{8.1}$$

which is a purely implicit scheme; other difference schemes can be constructed by evaluating the part of the operator in the inequality (8.1) which does not contain inertial terms in the time layer $t^\theta = t + \theta\tau$, $0 \leqslant \theta \leqslant 1$. For $\theta = 0$ we obtain an explicit scheme.

The existence and uniqueness of the solution of the inequality (8.1) is proved exactly the same as in Sect.4, since the operator which occurs because of the inertia forces is positive-definite.

Let us present a scheme to investigate the convergence as $\tau \to 0$. Having constructed the difference analog of the inequality (7.5), we see that the sequence of approximate solutions is bounded in $L^2(0, T; V)$ uniformly in τ, the sequence of approximations of the velocities $(u^{t+\tau} - u^t)/\tau$ is bounded in $L^2(0, T; L^2(\Omega))$. Imposing suitable smoothness requirements on the external effects, the shape of the stamp, and the condition for initial and boundary conditions to agree, we establish the boundedness of the acceleration approximation. Afterwards, by using the properties of the approximation, and the continuity of the forms in the inequality (8.1), we pass to the limit as $\tau \to 0$.

Let us emphasize (and Tremolières /10/ noticed this first) that the investigation of difference schemes in inequalities differs in principle from the investigation of schemes for equations since convergence does not here follow generally from the approximation and the stability.

9. Generalizations. As a number of authors has noted, the Coulomb friction law does not result in physically absurd results as, for instance, the unbounded growth of stresses in the absence of material flow; the following law /11/ is more real:

$$|\sigma_T| < f|\sigma_N| \ \text{and} \ |\sigma_T| < \tau_s \Rightarrow u_T^\cdot = 0 \tag{9.1}$$

$$|\sigma_T| = f|\sigma_N| \ \text{or} \ |\sigma_T| = \tau_s \Rightarrow \frac{u_T^\cdot}{|u_T^\cdot|} = -\frac{\sigma_T}{|\sigma_T|}$$

where $f = \text{const}$ or is a function of $|u_T^\cdot|$, $\tau_s = \text{const}$ or is a function of the strain and the strain rate intensities.

Reasoning analogous to that performed in Sect.2 yields the inequality

$$a(u^{t+dt}, w-u^{t+dt}) - L_*^{t+dt}(w-u^{t+dt}) + \int_{S_c} [f|\sigma_N| + (\tau_s - f|\sigma_N|)\theta_0(f|\sigma_N| - \tau_s)][|dw_T| - |du_T|]\,dS \geqslant 0 \tag{9.2}$$

where θ_0 is the Heaviside unit ("step") function.

The algorithm for the solution of this inequality is constructed exactly as above, the difference being that the additional constraint $|\sigma_T| \leqslant \tau_s$ is imposed on the tangential stress.

In conclusion, let us note that the variational approach to investigate the influence of friction in the problem of the contact between a deformable body and a rigid stamp was first applied in /12/, however, the formulation considered in /12/ possesses a numer of defects from the mechanics viewpoint: the Coulomb law relates the stresses and displacements (let us note that in individual cases the passage from velocities to displacements is actually possible /13/): the product $f|\sigma_N|$ is considered constant; the contact zone is assumed constant. These hypotheses permitted reducing the problem to the minimization of a nondifferentiable function without constraints, while, as shown in this paper, physically the meaningful formulation results in a sequence of minimization problems with constraints in the form of inequalities.

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